

Non-linear gravitational clustering of cold matter in an expanding universe: indications from 1D toy models

Michael Joyce^{1,2} and François Sicard¹

¹*Laboratoire de Physique Nucléaire et Hautes Énergies, Université Pierre et Marie Curie - Paris 6, CNRS IN2P3 UMR 7585, 4 Place Jussieu, 75752 Paris Cedex 05, France*

²*Laboratoire de Physique Théorique de la Matière Condensée, Université Pierre et Marie Curie - Paris 6, CNRS UMR 7600, 4 Place Jussieu, 75752 Paris Cedex 05, France*

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ABSTRACT

Studies of a class of infinite one dimensional self-gravitating systems have highlighted that, on the one hand, the spatial clustering which develops may have scale invariant (fractal) properties, and, on the other, that they display “self-similar” properties in their temporal evolution. The relevance of these results to three dimensional cosmological simulations has remained unclear. We show here that the measured exponents characterizing the scale-invariant non-linear clustering are in excellent agreement with those derived from an appropriately generalized “stable-clustering” hypothesis. Further an analysis in terms of “halos” selected with a friend-of-friend algorithm reveals that such structures are, statistically, virialized across the range of scales corresponding to scale-invariance. Thus the strongly non-linear clustering in these models is accurately described as a virialized fractal structure, very much in line with the “clustering hierarchy” which Peebles originally envisaged qualitatively as associated with stable clustering. If transposed to three dimensions these results would imply, notably, that cold dark matter halos (or even subhalos) are 1) not well modeled as smooth objects, and 2) that the supposed “universality” of their profiles is, like apparent smoothness, an artefact of poor numerical resolution.

Key words: gravitation; cosmology: large-scale structure of universe;

1 INTRODUCTION

The much acclaimed successes of the Λ CDM cosmology in matching many observations concern essentially its homogeneous limit and the linear regime in which perturbations to homogeneity are small. The success of the model in accounting for the numerous observations which probe the non-linear regime, where density fluctuations are large, is much more uncertain. An important consideration in this respect is the great difficulty of calculating the model’s predictions in this regime. Even in the idealized limit in which clustering arises from gravity alone, predictions reside solely on numerical simulations. The latter have, despite impressive increases in their size and sophistication, still quite limited spatial resolution. Further, essentially because there are no non-trivial analytical benchmarks which they can be tested against, it remains unclear to what extent their results are conditioned by these resolution limits. In this paper we explore what might be learnt about the nature of non-linear gravitational clustering from the study of a class of one dimensional (1D) models, which have the interest of offering, even with modest computer resources, very much greater spatial resolution and exact numerical integration. An early

study in this spirit is that of Melott (1982, 1983) which used such a 1D model to explore clustering in hot dark matter cosmologies. Rouet et al. (1990) derived and studied a slightly different model to the most naive 1D version of clustering in an Einstein de Sitter (EdS) universe considered by Yano & Gouda (1998); Aurell & Fanelli (2002); Aurell et al. (2003). Valageas (2006) and Gabrielli et al. (2009) have studied the version without expansion, while Miller et al. (Miller & Rouet 2002, 2006; Miller et al. 2007; Miller & Rouet 2010) have reported results on all three of these models. Despite these studies it remains unclear, however, whether a good, and really useful, analogy can be made with 3D cosmological simulations. Generalizing the results of Yano & Gouda (1998) and Gabrielli et al. (2009), we confirm here the very strong qualitative similarities in the temporal development of clustering to that in the 3D case. Further we show the scale invariant properties of the clustering in real space, emphasized and analyzed by Miller & Rouet (2002, 2006); Miller et al. (2007), can in fact be well explained within an analytical framework similar to one proposed long ago for the 3D case (Peebles 1974; Davis & Peebles 1977; Peebles 1980). We derive the appro-

appropriate 1D version of this generalized “stable clustering” prediction for the correlation exponents, and show it to provide a very good approximation to the numerical results. We also show that, in a precise sense, the strongly non-linear clustering can be properly described as a *virialized* fractal hierarchy, in line with what was originally anticipated qualitatively in three dimensions. We argue finally for the extrapolation of these conclusions about the qualitative nature of clustering to the relevant 3D case, and consider its implications, in particular for what concerns the nature and properties of “halos” in cold dark matter cosmologies.

2 1D VERSIONS OF COSMOLOGICAL N-BODY SIMULATIONS

Dissipationless cosmological N-body simulations (for a review see, e.g., Bagla (2005)) solve numerically the equations

$$\frac{d^2 \mathbf{x}_i}{dt^2} + 2H \frac{d\mathbf{x}_i}{dt} = -\frac{Gm}{a^3} \sum^J \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}, \quad (1)$$

where \mathbf{x}_i are the comoving particle coordinates, $a(t)$ is the appropriate scale factor for the cosmology considered, with Hubble constant $H(t) = \dot{a}/a$. The superscript ‘J’ in the sum, which runs in practice over the infinite system constituted by periodic copies of a cube containing N particles, indicates that the (badly defined) contribution of the mean mass density (which sources the Hubble expansion) is subtracted. We consider here simply the 1D system obtained by replacing the 3D Newtonian force term by the analogous 1D expression, derived starting from the 1D Poisson equation which gives a force between particles in one dimension (or, equivalently, infinite parallel sheets embedded in three dimensions) independent of separation. We thus consider the equations

$$\frac{d^2 x_i}{dt^2} + 2H \frac{dx_i}{dt} = -\frac{g}{a^3} \lim_{\mu \rightarrow 0} \sum_{j \neq i} \text{sgn}(x_i - x_j) e^{-\mu |x_i - x_j|}, \quad (2)$$

where the limiting procedure used in the sum is simply a convenient way to explicit the subtraction of the background (KieSSLing 2003), and g is the coupling constant (with $g \equiv 2\pi\Sigma G$ for sheets of surface mass density Σ). Initial conditions are generated, just as in 3D cosmological simulations, by applying appropriate small perturbations to particles initially on an infinite perfect lattice. If the displacement from its initial lattice site of particle i is u_i it can be shown rigorously (Gabielli et al. 2009) that the net force on the particle, until it crosses another one, is exactly proportional to $u_i - \langle u \rangle$, where $\langle u \rangle$ is the average displacement. The equations of motion up to the time at which particles cross become

$$\ddot{u}_i + 2H\dot{u}_i = \frac{2gn_0}{a^3} u_i \quad (3)$$

where n_0 is the mean particle density (and we take $\langle u \rangle = 0$). When the *particles* in our model cross one another the force changes simply by $\pm 2g$. However, since in 1D a crossing of particles is equivalent, up to exchange of labels, to an elastic “collision” between them, one can instead consider (as we are not interested in the labels of the particles) a system of particles which bounce elastically and follow Eq (3) *at all times* between “collisions”. Finally, by a transformation

of the time coordinate to $\tau = \int dt/a^{3/2}$, and appropriate choice of units of τ , we can rewrite these equations as

$$\frac{d^2 u_i}{d\tau^2} + \Gamma \frac{du_i}{d\tau} = u_i, \quad (4)$$

i.e., as those of a set of damped inverted harmonic oscillators. We note that these equations coincide simply with those for displacements of *fluid* elements in the Zeldovich approximation (see, e.g. Buchert (1992)), which is in fact exact up to “shell crossing” for 1D perturbations. The same system can thus be derived as a simple analytical continuation of this approximation at shell crossing (Yano & Gouda 1998; Aurell et al. 2003).

For a generic cosmology Γ in Eq. (4) is a non-trivial function of τ , but in the specific case of an EdS cosmology (for which $a \propto t^{2/3}$) it is a constant, $\Gamma = 1/\sqrt{6}$ (and $\tau \propto \ln a$). The model derived and studied by Rouet et al. (1990) corresponds instead to the case $\Gamma = 1/\sqrt{2}$, while the case $\Gamma = 0$ (i.e. static limit) has also been studied by several authors (see references given above). We note that, despite the fact that we obtained the value $\Gamma = 1/\sqrt{6}$ above, this is not necessarily *the* more appropriate value to consider for our study: in the derivation just given the 3D Hubble law has been imposed *by hand*. This has been done because the 1D Hubble law (which would arise if one follows fully the analogy with the 3D derivation starting from physical coordinates) gives a completely different qualitative behavior — collapse in a finite time — which has no relevance to 3D cosmology. Thus in order to “mimic” EdS cosmology, one could equally take the functional form of the corresponding Hubble law, but leave its normalization as a free parameter. This leads to the Eqs. (4) with Γ an undetermined constant. These equations then simply define a simple class of toy models, which may be useful for *qualitative* understanding of the 3D problem.

Given initial displacements and velocities, Eqs. (4) can trivially be integrated exactly between particle collisions when Γ is a constant. To determine the time of the next collision (and which pair of particles it involves) requires then only the solution of algebraic equations. The numerical integration can therefore be performed “exactly”, i.e., up to machine precision. As described by Noullez et al. (2003) the integration can be sped up optimally using a “heap” structure. All the results presented here are for systems with $N = 10^5$ particles (and periodic boundary conditions).

In making the analogy with 3D simulations, which aim to reproduce the *collisionless* limit of gravitational clustering and include a smoothing of the singularity in the 3D Newtonian force for this reason, it is to be noted that, on the time scales we will consider, these 1D systems, without force smoothing, can be expected to represent extremely well this limit. Studies of collisional relaxation in finite 1D self-gravitating systems (see Joyce & Worrakitpoonpon (2010), and references therein) have shown that it is very suppressed compared to that in 3D systems, a typical relaxation time for a virialized system of number density n being $\sim (10^4 - 10^6)N/\sqrt{gn}$. Using this estimate in the simulations reported below, it is simple to show that even the relaxation time for the smallest and densest clusters is much longer than the duration of the simulation. The reason for this relative suppression of this relaxation is that, in one dimension, there is no analogy to 3D 2-body relaxation: the “collisions”

of particles we have discussed above do not contribute to the usual collision term in the Boltzmann equation. Indeed it is for this reason that particle crossings and “collisions” are equivalent.

3 COMPARISON WITH SIMULATIONS IN 3D: SELF-SIMILARITY

Various studies (see references above) of these models starting from close to uniform initial conditions have noted the striking qualitative similarity of the clustering to that in three dimensions: one observes the formation of structures (i.e. overdense regions) first at small scales and then at ever larger scales. Let us consider first more closely the temporal character of this evolution. It is canonical and instructive in the study of cold dark matter clustering in the universe to consider evolution from initial conditions characterized by power spectrum (PS) $P(k) \sim k^n$ (with n a constant), and velocities prescribed by the growing mode of linear perturbation theory. Although the “real” cosmological power spectrum (in currently favoured *cold* dark matter models) is not exactly power-law it may be well approximated as such, with a slowly varying exponent, between $n = -1$ and $n = -3$ over the relevant range (see, e.g. Peebles (1993)). The simulation of such “scale-free” initial conditions is expected to give rise to simple behaviours (which one might hope to understand analytically). In particular, for $-d < n < 4$ (where d is the spatial dimension) one expects theoretically (Peebles 1980) a process of hierarchical structure formation which

- is driven by linear amplification of the initial fluctuations. For power law initial conditions and the models we are considering, it is simple to show that this corresponds to the scale of non-linearity (at which perturbations are of order one) which grows in proportion to

$$R_s(t) = \exp\left(\frac{2\lambda_+(\Gamma)}{1+n}\tau\right), \quad (5)$$

where $\lambda_+(\Gamma) = -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 + 1}$ is the root of the characteristic equation of Eq. (3) associated to the growing mode of linear theory;

- tends asymptotically to “self-similar” behaviour in the whole non-linear regime, i.e., the temporal evolution of all correlation properties can be obtained by a rescaling of the spatial coordinates, in proportion to the same function $R_s(t)$.

Extending analyses in Yano & Gouda (1998) and in Gabrielli et al. (2009) we have studied a range of values of initial PS ($n = 0, 2, 4$) and models ($\Gamma = 0, 1/\sqrt{6}, 1/\sqrt{2}$). The upper panel in Fig. 1 shows the measured reduced two point correlation function $\xi(x, t)$ as a function of $x/R_s(t)$, while the lower panel shows the (dimensionless) $\Delta^2 = kP(k)$ as a function of $kR_s(t)$, for the case of an initial PS with $n = 2$ and a model with $\Gamma = 1/\sqrt{6}$. The superposition of the rescaled PS at small k in the lower panel shows the validity of linear theory, while the superposition in the same plot at larger k , and of $\xi(x)$ in the upper panel over a wide range, shows the development of the corresponding self-similarity in the non-linear regime. We observed the same behaviours

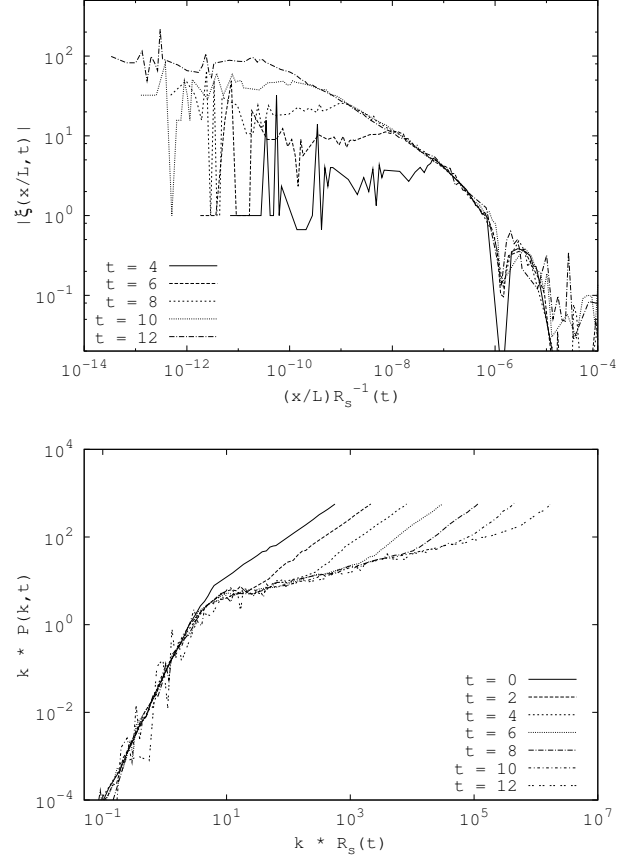


Figure 1. The two point correlation function (upper panel) and power spectrum (lower panel) as a function of rescaled spatial variables, for the $\Gamma = 1/\sqrt{6}$ model and initial conditions with $n = 2$ (i.e. EdS cosmology). The “time” variable $t \equiv \log a$ (where $a = 1$ initially). L is the (periodic) box size.

in all cases, except for $n = 4$. In the latter case it turns out that, as expected, linear theory breaks down at small k , but nevertheless self-similarity is observed in the non-linear regime.

In three dimensions such self-similar behaviour has been observed numerically to apply in all cases which have been simulated. Specifically in EdS universes it has been shown to apply in the range $-3 < n \leq 1$ (see, e.g., Efstathiou et al. (1988); Colombi et al. (1996); Jain & Bertschinger (1998); Smith et al. (2003)), while in the static universe limit it has been shown to apply for $n = 0$ and $n = 2$ (Baertschiger et al. 2007, 2008). As in all these 3D cases, a good power-law fit can be obtained (as in Fig. 1) to the correlation function, $\xi(r) \sim r^{-\gamma}$, and to the power spectrum ($k^d P(k) \sim k^\gamma$) in the range of non-linear self-similar clustering. The exponents γ we measure in our 1D simulations, which are reported below in further detail, show the same qualitative dependence on n as in 3D: in the expanding case γ increases as n does so (see, e.g., Smith et al. (2003)), while in the static case one observes a much smaller exponent which does not depend sensibly on n (Baertschiger et al. 2007, 2008).

4 EXPONENTS OF SCALE INVARIANT CLUSTERING

Compared to 3D the power law behaviours we observe in the 1D models extend over a much larger range of scales, simply because of the much greater dynamical range accessible. In Fig. 1, for example, such behaviour extends over roughly four decades in scale, compared to a single decade in 3D studies (e.g. Smith et al. (2003)). Such spatial resolution makes it possible to robustly determine whether such behaviour is indicative of scale invariant (i.e. fractal-type) properties in the distributions, using tools such as box-counting methods to measure the spectrum of “generalized dimensions” (see, e.g. Gabrielli et al. (2005)). In a recent study of these 1D models with initial conditions like those we are studying here, Miller & Rouet (2010) have used such methods to analyze this clustering and have shown that, like in their previous studies of slightly different initial conditions in Miller et al. (2007), they indeed find clear evidence for scaling behaviours indicative of fractal clustering in this regime. We have performed much of the same analysis on our simulations, and find results very consistent with those reported in Miller & Rouet (2010): robustly measured “generalized dimensions” in the range of scales where the power law behaviour in the correlation function indicates scale invariance, and in particular a value of the “correlation dimension” $D_2 \approx 1 - \gamma$ as expected.

We show now that the exponent γ measured from $\xi(x, t)$ can be very well accounted for, in the expanding models, using both the self-similarity of the evolution in time, and directly observable “stable clustering” behaviour in the system at sufficiently small scales. To do so let us consider the temporal evolution of the lower cut-off, x_{min} say, at which the self-similar scaling breaks down. Just as can be seen in Fig. 1 we observe in all our simulations that $\xi(x, t)$ breaks from the power-law at this scale and become approximately flat, reaching a maximum value ξ_{max} . (In 3D simulations the effects of force smoothing make it impossible to follow in this way the development of self-similarity at small scales). We observe to a very good approximation, in our simulations with $\Gamma \neq 0$, that

$$\xi_{max} \propto x_{min}^{-1} \sim \exp(-\epsilon\tau) \quad (6)$$

where ϵ is a constant *which depends only on Γ* (and not on n). Further we observe that the upper cut-off to power-law behaviour in the correlation function, x_{max} , occurs (as in three dimensions) at a fixed amplitude in ξ slightly larger than unity, i.e., roughly at the non-linearity scale [defined by $\xi(x_{NL}) = 1$]. We therefore have, from self-similarity, that $x_{max} \propto R_s(t)$. It follows that the exponent γ , at sufficiently long times, should be given by

$$\gamma = \frac{\epsilon}{\epsilon + \frac{2}{1+n}\lambda_+(\Gamma)}. \quad (7)$$

To determine the exponent fully we need only determine how ϵ in (6) depends on Γ . The relation (6) indicates the relevance of “stable clustering”: if the correlation *up to this scale* x_{min} arises from structures which do not evolve (macroscopically) when considered in spatial coordinates rescaled in proportion to x_{min} , one obtains such a behaviour. In three dimensions such a behaviour has been proposed long ago (Peebles 1974; Davis & Peebles 1977) as possibly valid *in the strongly non-linear regime*, and used

to obtain a prediction for the exponent γ in an EdS universe. It amounts to supposing that if non-linear structures behave essentially as isolated systems (i.e. if the tidal forces exerted by all mass external to them are negligible), they will be expected to virialize and remain stable in *physical* coordinates. In the 1D models we are considering the equations of motion have not been derived starting from physical coordinates, but such “stable clustering” nevertheless has a clear meaning¹. Indeed in the 1D model the meaning of “isolation” of a subsystem may be given more exactly than in 3D: if particles of a given subsystem do not cross particles outside the system, the corresponding system indeed evolves independently from the rest of the “universe” (i.e. tidal gravitational forces vanish in 1D). To see what evolution is then predicted, it is appropriate to rewrite the equations of motion for such a finite isolated subsystem using the particle labelling in which particles pass through one another. It is straightforward to show that the equation of motion for particles belonging to such a subsystem may then be written (in the units we have adopted in which $g = 1/2n_0$)

$$\frac{d^2 x_i}{d\tau^2} + \Gamma \frac{dx_i}{d\tau} = g(N^>(i) - N^<(i)) + x_i \quad (8)$$

where x_i is the position of the particle i *with respect to the centre of mass of the system*, and $N^>(i)$ ($N^<(i)$) is the number of particles (in the subsystem) on the right (left) of the particle i . Unsurprisingly this is just the equation of motion for a *finite* 1D self-gravitating system in an infinite space, with an additional damping term arising from the expansion and a term from the subtracted mean density. We can define the conserved energy E associated with the two terms on the right hand side, and have then

$$\frac{dE}{d\tau} = -2\Gamma K \quad (9)$$

where K is the total kinetic energy, and $E = K + U_{grav} + U_{bg}$ with $U_{grav} = \frac{1}{2} \sum_{i,j} g |x_i - x_j|$ and $U_{bg} = \sum_i x_i^2/2$. Now, if such an isolated subsystem is significantly *overdense* (i.e. its mean density n_s is much greater than n_0), the time scale associated with the dynamics of the first term on the right hand side $\tau_{grav} \sim \sqrt{n_0/n_s}$ is much shorter than that associated with the other two terms, and the term U_{bg} is negligible compared to U_{grav} . We expect this to allow the use of an adiabatic approximation to (9), in which E and K are replaced by their values averaged on this shorter time scale, $\langle E \rangle$ and $\langle K \rangle$. On these time scales, however, we expect the corresponding virial relation to apply, i.e., $\langle 2K \rangle = \langle U_{grav} \rangle$. It follows that $\langle E \rangle = 3\langle K \rangle$, and then, using Eq. (9), that

$$\langle K \rangle \propto \exp\left(-\frac{2\Gamma\tau}{3}\right) \propto \langle U_{grav} \rangle. \quad (10)$$

Given that U_{grav} is linearly proportional to the size of the subsystem, the size of such a virialized subsystem will be expected to scale in the same way. Thus, if the behaviour in (6) is indeed due to such stable virialized structures, $\epsilon = 2\Gamma/3$. Substituting this relation in (7) gives

$$\gamma = \frac{2\Gamma(n+1)}{\Gamma(2n-1) + 3\sqrt{\Gamma^2 + 4}}. \quad (11)$$

¹ We note that Yano & Gouda (1998) give an incorrect generalization to 1D of the stable clustering hypothesis, assuming that stability will be attained in the 3D physical coordinates.

initial PS	$\Gamma = 0$ (thy)	$\Gamma = 0$ (sim)	$\Gamma = 1/\sqrt{6}$ (thy)	$\Gamma = 1/\sqrt{6}$ (sim)	$\Gamma = 1/\sqrt{2}$ (thy)	$\Gamma = 1/\sqrt{2}$ (sim)
$n = 0$	$\gamma = 0$	$\gamma = 0.18 \pm 0.03$	$\gamma = 1/7$	$\gamma = 0.14 \pm 0.02$	$\gamma = 1/4$	$\gamma = 0.25 \pm 0.02$
$n = 2$	$\gamma = 0$	$\gamma = 0.18 \pm 0.03$	$\gamma = 1/3$	$\gamma = 0.35 \pm 0.02$	$\gamma = 1/2$	$\gamma = 0.50 \pm 0.02$
$n = 4$	$\gamma = 0$	$\gamma = 0.15 \pm 0.05$	$\gamma = 5/11$	$\gamma = 0.43 \pm 0.01$	$\gamma = 5/8$	$\gamma = 0.62 \pm 0.01$

Table 1. Values of the exponent $\gamma(n, \Gamma)$ characterizing the power-law range in $\xi(x)$, as measured in simulations (“sim”) and as predicted theoretically (“thy”) by (11), which gives $\gamma = (n+1)/(n+7)$ for $\Gamma = 1/\sqrt{6}$, and $\gamma = (n+1)/(n+4)$ for $\Gamma = 1/\sqrt{2}$.

We note that, using the above arguments, a straightforward generalization of the usual 3D stable clustering prediction (for an EdS cosmology) to the class of cosmological backgrounds corresponding to any (constant) Γ may be obtained. The result is²

$$\gamma_3 = \frac{6\Gamma(n+3)}{\Gamma(2n+5) + \sqrt{\Gamma^2 + 4}} \quad (12)$$

which indeed coincides with the usual EdS prediction, $\gamma = 3(3+n)/(5+n)$, for $\Gamma = 1/\sqrt{6}$.

Shown in Table 1 is a comparison of the prediction (11) with the corresponding exponents measured in our simulations. The agreement is excellent for the expanding cases, and indeed in these cases the value ϵ measured directly agrees well with the predicted $\epsilon = 2\Gamma/3$ ³. In the static case the prediction leads to the badly defined result $\gamma = 0$, while we observe a small exponent $\gamma_0 \approx 0.16$ which does not appear to depend on n . We observe in this case that, rather than (6), we have an x_{min} which remains approximately constant (i.e. the expected $\Gamma = 0$ limit), but a ξ_{max} which grows. Clearly the stable clustering hypothesis is indeed only an approximation, to which corrections (due to the fact that structures are not, of course, strictly isolated) become most important, as one would expect, when we go to the limit $\Gamma \rightarrow 0$ in which virialized structures no longer decrease in size.

5 HALOS AND VIRIALIZATION IN THE SCALE INVARIANT REGIME

Let us now consider the observed clustering in terms of “halos”, in analogy to how this is done in cosmological simulations. We thus use a friend-of-friend algorithm, with a variable linking length Λ (which we define in units of the initial lattice spacing ℓ). In 1D this is equivalent to separating the system into structures separated by voids of size greater than or equal to Λ . In the range of scale-invariant non-linear clustering, i.e., for $x_{min}/\ell < \Lambda < x_{NL}/\ell$, we observe that the algorithm selects out structures with a broad range of sizes from our simulations, peaked around a value which scales with Λ . However, the fact that the distribution of sizes is quite large, while the range of scale invariance

in any simulation is of course quite limited, make it difficult (even in 1D) to use such halos as a tool for characterizing this regime. We thus analyse here the case of an initial PS with $n = 4$ in which this range is greatest — a little more than four orders of magnitude in scale between $x_{min}/\ell \approx 10^{-2.5}$ and $x_{NL}/\ell \approx 10^{1.5}$ in the most evolved configuration.

Shown in Fig. 2 is the measured distribution of virial ratio as a function of Λ in this simulation, i.e. the distribution of the values of the ratio $2K/U_{grav}$ measured directly in each halo, taking the velocities with respect to the centre of mass of the corresponding halo. We observe an approximately stable distribution *peaked about unity* in the range of Λ where the structures selected out fall mostly in the length scales of scale invariance, while at the smallest scales no such tendency is observed. On the contrary the virial ratio tends to be very large, which can be interpreted as due to the fact that the algorithm is selecting structures inside the scale x_{min} : if the distribution at this scale is constituted of approximately smooth virialized structures of size x_{min} , parts of these structures will have a high virial ratio (because K scales more slowly with size than U_{grav} in a smooth structure).

We have studied quantitatively the stability of these measured distributions in the range of scale invariant clustering using a nonparametric Kolmogorov-Smirnov (K-S) test. The K-S statistic quantifies a distance between the empirical distribution functions of the different samples, and the null distribution is calculated under the null hypothesis that the samples are drawn from the same distribution. Calculating the corresponding p -values of this test, we find that the null hypothesis is indeed not rejected in the range of scale-invariance, while, on the other hand, it is clearly rejected outside this range.

Our interpretation of these results is that one can effectively decompose the distribution into a collection of structures which are, approximately, virialized. This is true provided these structures are defined at any scale in the range of scale-invariant clustering.

6 DISCUSSION

In three dimensions the stable clustering hypothesis has been used widely as a reference point, and indeed much used phenomenological models such as the formalism proposed by Peacock & Dodds (1996) incorporate it into the modeling of the non-linear power spectrum obtained from numerical simulations. However, in parallel, results of major simulations of power law initial conditions (e.g. Efstathiou et al. (1988); Smith et al. (2003)) led to the conclusion that, although the measured exponents in the correlation function showed a behaviour roughly consistent with its predictions

² Virialization in three dimensions leads to $\langle E \rangle = -\langle K \rangle$, so that $\langle U_{grav} \rangle \propto \exp(2\Gamma\tau)$. It follows that $x_{min} \propto \exp(-2\Gamma\tau)$, and then, with $\xi_{max} \propto x_{min}^{-3}$ and the 3D expression for $R_s(t)$, the result follows.

³ For the case $\Gamma = 1/\sqrt{6}$ our measured exponents agree with those reported by Yano & Gouda (1998). Our stable clustering result $\epsilon = 2\Gamma/3$ explains also the exponent measured by Aurell et al. (2003) directly for an isolated structure.

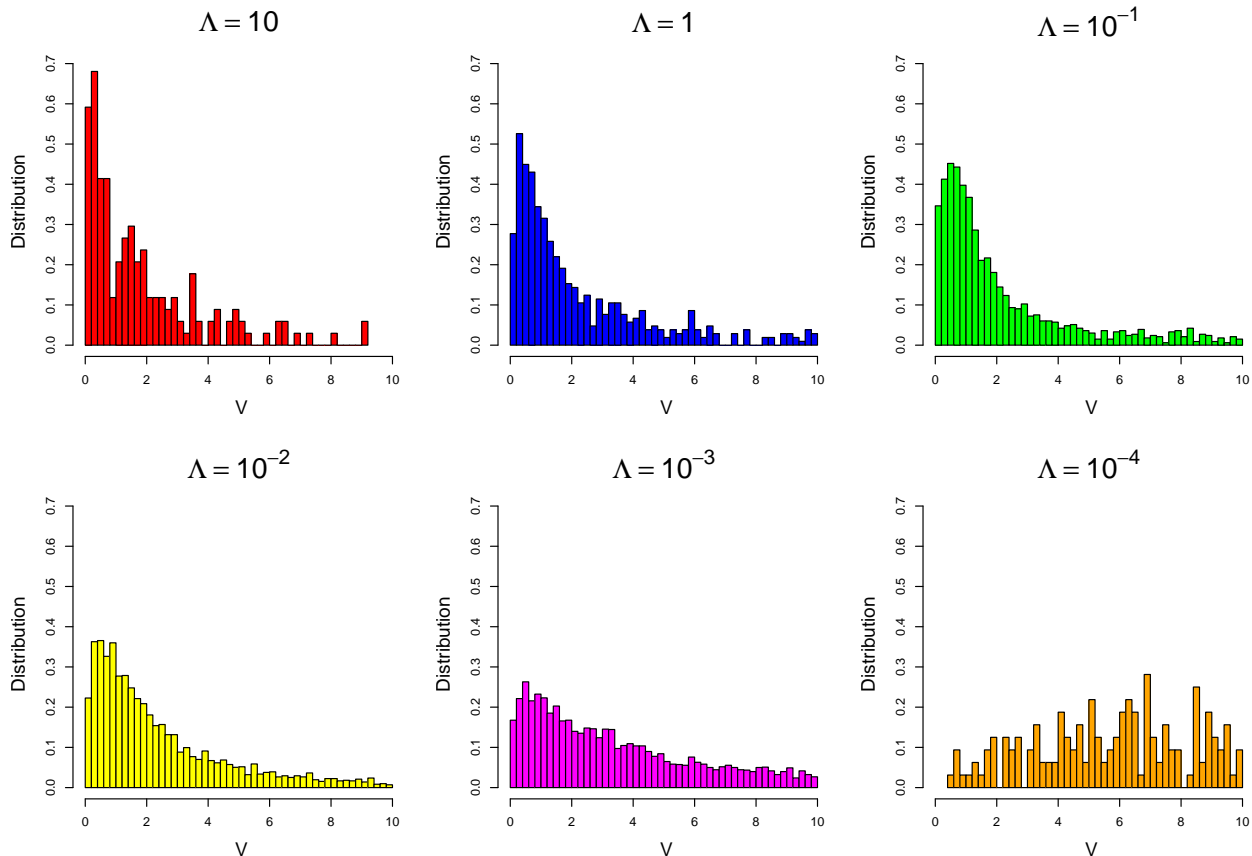


Figure 2. Normalized distribution of the virial ratios of halos selected with the indicated values of Λ (linking length in units of lattice spacing), for the case $n = 4$ and $\Gamma = 1/\sqrt{6}$ at $t = \ln a = 22$.

(albeit with some significant deviations (Smith et al. 2003)), there was no evidence for the “clustering hierarchy” which Peebles had argued would be associated with it (see, e.g. Peebles (1980)). Efstathiou et al. (1988), notably, explicitly excluded such a “fractal” description and found evidence instead for the validity of a description in terms of “smooth non-linear clumps”. These latter are the precursors of the “halos” of halo models, which have become the standard phenomenological description of the matter distribution in cold dark matter cosmologies (for a review, see, e.g. Cooray & Sheth (2002)). The matter distribution is then approximated as a collection of isolated (and thus virialized) spherical structures with smooth density profiles. Further, on the basis of extensive numerical study following that of Navarro et al. (1997), the latter are widely believed to be characterized well by “universal” exponents (i.e. independent of initial conditions and cosmology).

In one dimension the strongly non-linear regime is truly scale-invariant (for power law initial conditions) in a range which grows monotonically in time. The associated distribution of matter is *intrinsically* lumpy or grainy down to the lower cut-off scale x_{min} : indeed the very meaning of such *scale invariance* is that there are no characteristic scales available to define smoothness. To illustrate this we show in Fig. 3 the spatial density at various levels of detail in a typical evolved configuration. Such distributions are most naturally described with the instruments of frac-

tal (or, more generally, multi-fractal) analysis developed for this purpose in condensed matter and statistical physics (see e.g. Gabrielli et al. (2005)). As we have just illustrated in the previous section, halo type descriptions may, of course, also be employed to describe them. The halos so defined are, however, very different to those described by 3D halo models: these 1D halos are not smooth. Further have they no intrinsic size themselves, but are defined only with reference to an arbitrary chosen scale. The study of the virial ratios we have presented indicates, however, that such halos can be considered as entities with a dynamical relevance, as they show a clear tendency to have a virial ratio of order unity (which is the behaviour of an isolated structure). The clustering in the non-linear regime can thus be considered as a concrete realization of the qualitative picture of a “clustering hierarchy” originally envisaged by Peebles (e.g. Peebles (1980)) as resulting from stable clustering. The stable clustering hypothesis we have described above, however, is actually subtly different from the original one: we assumed only that stable clustering applies below the scale x_{min} marking the lower cut-off to the scale invariance, and not necessarily to the strongly non-linear regime as a whole. Thus we assumed only that stable clustering applies at an ultraviolet scale *fixed by the resolution of the simulation* (or, physically, by the scale at which the very first structures form). The “statistical virialization” we have observed using the halo

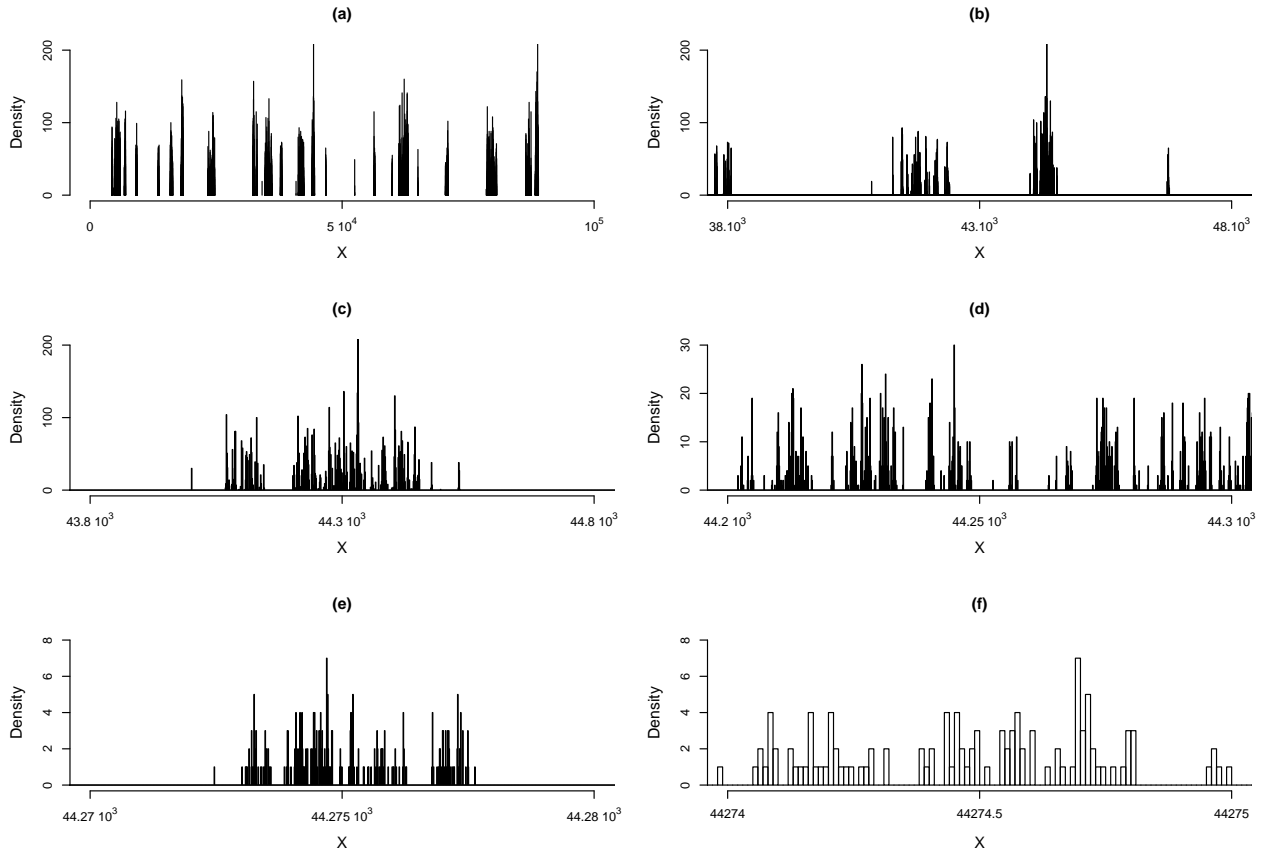


Figure 3. Density field obtained at $t = \log a = 14$ starting from an initial condition with $n = 2$, in the “EdS” model with $\Gamma = 1/\sqrt{6}$. The first panel shows the whole box (i.e. length units are such that $L = N = 10^5$), while each subsequent panel shows a spatial “zoom” on a region of size *one tenth* that shown in the previous panel. The resolution has been increased (i.e. the bin size has been decreased) in the last three plots in order to reveal the clumpy nature of the distribution at these scales

analysis, on the other hand, applies at scales above x_{min} and across the range of the scale invariant clustering.

There are clearly two possible conclusions one can draw from this analysis:

- A. These 1D models produce non-linear clustering which is qualitatively different in its nature to that in 3D, or
- B. The spatial resolution in 3D simulations up to now has been too limited to reveal the nature of clustering in cold dark matter cosmologies, which is correctly reflected (qualitatively) in the 1D simulations.

We believe that, despite the impressive computational size and sophistication of 3D cosmological simulations, conclusion B may well be the correct one. The very largest modern studies in a cosmological volume access roughly two decades in scale in the non-linear regime⁴ while reference studies in the literature of power law initial conditions in

EdS cosmology (Efstathiou et al. 1988; Smith et al. 2003) measure the crucial power-law behaviour in the correlation function (or the PS) over at most one decade. If we were to perform our 1D simulations at comparable resolution to large cosmological simulations like Smith et al. (2003), we would certainly have great difficulty in establishing the scale-invariant nature of the strongly non-linear clustering arising from power law initial conditions. Although halos defined exactly as in three dimensions might look clumpy, an approximately smooth profile could be determined for them if they were averaged (as they can be in three dimensions when spherical symmetry is assumed). Higher resolution 3D simulations of smaller regions have shown over the last decade that there is in fact much more substructure (“subhalos”) inside halos than was originally anticipated (see, e.g., Moore et al. (1999); Diemand et al. (2005); Goerdts et al. (2007)), and have even more recently described several levels of such substructure (“subhalos of subhalos”, see e.g. Diemand et al. (2008); Springel et al. (2008); Stadel et al. (2009)). Diemand et al. (2008) even use the term “fractal” to describe (qualitatively) the real space structures, while

⁴ For example, the “millennium” simulation (Springel et al. 2005) of the Λ CDM cosmology, has, at redshift zero, $\xi = 1$ at about $2h^{-1}\text{Mpc}$, an initial mean interparticle $\ell \approx 250h^{-1}\text{kpc}$ and a force smoothing $\epsilon = 5h^{-1}\text{kpc}$. It is usually supposed that it is the latter which sets the lower bound on resolution, but we note that it has been shown by several studies (see Melott et al.

(1997); Romeo et al. (2008); Joyce et al. (2009)) that, at the very least, precision may be compromised below the scale ℓ .

Zemp et al. (2009) describe the structure of halos in phase space as “intrinsically grainy”. We note that other authors (see, e.g., Valageas (1999); Gaite (2007)) have previously argued for similar conclusions on the basis of analyses of 3D simulations.

Let us consider nevertheless one possible consideration in favour of (the more conservative) conclusion A. In the expanding (i.e. damped) 1D models, the stable clustering prediction (11) fits the measured exponents extremely well. Early 3D studies for EdS cosmologies (e.g. Efstathiou et al. (1988)) measured exponents roughly consistent with the stable clustering prediction, but later studies (e.g. Smith et al. (2003)) have found significant disagreement. This disagreement is attributed to physical mechanisms which cause the fundamental assumption of stability to be violated — by the evident fact that *there are* interactions between “halos”, which can even lead to their merging into single structures. We have noted that in one dimension tidal forces vanish, and structures can interact only when they actually physically cross one another. While merging may occur, it may be that it is a less efficient process than in three dimensions. Thus the excellent agreement in the 1D models compared to EdS may perhaps be attributed to the fact that these models probably represent poorly the role of such physical effects. The essential question, however, is not whether these effects play a role and can lead to deviations from stable clustering, but whether such effects can *qualitatively* change the nature of clustering, destroying scale invariance by smoothing out the distribution *on a scale related to the upper cut-off* to scale invariance. Our study of the case $\Gamma = 0$ suggests that the answer is negative. The prediction of stable clustering does not work in this case, and like in three dimensions, one obtains a small value of the exponent which does not sensibly depend on n . The physical reasons why the exponent is close to, but different to, the stable clustering prediction are a priori the ones just cited. The analysis of Miller et al. of this case, which we have rechecked and confirm, finds nevertheless that the distribution is scale invariant. Further, as we have mentioned, the lower cut-off x_{min} remains constant as in the stable clustering hypothesis, of order the initial lattice spacing (and unrelated to the upper cut-off).

These results on 1D models suggest directions for 3D investigations which might establish definitively the correctness of conclusion B. We note, for example, that the 1D models lead one to expect that the exponents derived phenomenologically to characterize the highly non-linear density field inside smoothed halos (i.e. the “inner slope” of halos) should be closely related to the exponent γ determined from the correlation function. Indeed — in the approximation of a simple fractal behavior in the strongly non-linear regime, which the spectrum of multi-fractal exponents measured in Miller & Rouet (2010) suggests should be quite good — the mean density about the centre of such halos will decrease just as about any random point, i.e., with the *same* exponent γ . Despite the contradiction with the widely claimed “universality” of such exponents in halos profiles, such a hypothesis cannot currently be ruled out, as the determination of such exponents is beset by numerical difficulties (arising again from the limited resolution of numerical simulations). In a study of halo profiles obtained from power law initial conditions Knollmann et al. (2008) show explicitly that the results for the halo exponents depend greatly

on what numerical fitting procedure is adopted. While one procedure gives “universality” (i.e. exponents independent of n), a different one favors clearly steepening inner profiles for larger n . Indeed we note that the numerical values for the inner slopes obtained by Knollmann et al. (2008) are, for the larger n investigated, in quite good agreement with the exponent predicted by stable clustering.

Our results here are for power law initial conditions, but we could equally use the model to study both initial conditions and an appropriately modified time-dependent damping rate mimicking the Λ CDM model in one dimension. The exact scale invariance in the strongly non-linear regime would certainly be broken in this case, and the various characteristic scales introduced will be imprinted on the clustering. There would thus not be a single correlation exponent, but a slowly varying one. One would certainly expect, however, the qualitative nature of the clustering to be unchanged, just as in three dimensions clustering appears to be qualitatively the same in “scale-free” cold dark matter cosmologies and Λ CDM. Further, our considerations here are strictly relevant only to dissipationless *cold* dark matter simulations. If the initial conditions are “warm” or “hot”, or if other non-gravitational interactions are turned on, the associated physical effects will tend to smooth out the matter distribution up to some scale (and thus destroy the scale invariance up to this scale). Nevertheless, if the conclusion B is correct even for this idealized case, it is likely to have very important observational implications relevant to testing standard cosmological models. Intrinsically clumpy or grainy halos lead, for example, to very different predictions for dark matter annihilation (see, e.g. Goerdt et al. (2007); Diemand et al. (2008); Afshordi et al. (2010))). Concerning the compatibility of such a distribution of cold dark matter with observations of the distribution of visible matter at sub-galactic scales — which would be expected to lead to an amplified “missing satellite” problem — we note that recent studies such as Wadepuhl & Springel (2010) show that solid conclusions in this respect will be reached only on the basis of a much improved understanding of the processes involved in star formation. At larger scales the possible link to the striking power-law behavior which characterizes galaxy correlations over several decades in scale (see, e.g., Peebles (1974); Sylos Labini et al. (1998); Masjedi et al. (2006)) — which was the motivation for original work using stable clustering to explain such behaviour from power law initial conditions (Peebles 1974) and is naturally interpreted as indicative of underlying scale invariance in the matter distribution (see, e.g. Sylos Labini et al. (1998); Gaite (2007); Antal et al. (2009)) — is intriguing, and will be discussed elsewhere.

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